

# Recovering Differential Operators with Nonlocal Boundary Conditions

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**Abstract.** Inverse spectral problems for Sturm-Liouville operators with non-local boundary conditions are studied. As the main spectral characteristics we introduce the so-called Weyl-type function and two spectra, which are generalizations of the well-known Weyl function and Borg's inverse problem for the classical Sturm-Liouville operator. Two uniqueness theorems of inverse problems from the Weyl-type function and two spectra are presented and proved, respectively.

**Key words:** differential operators; nonlocal boundary conditions; inverse spectral problems

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**1. Introduction.** Consider the differential equation

$$-y'' + q(x)y = \lambda y, \quad x \in (0, T), \quad (1)$$

and the linear forms

$$U_j(y) := \int_0^T y(t) d\sigma_j(t), \quad j = 1, 2. \quad (2)$$

Here  $q(x) \in L(0, T)$  is a complex-valued function,  $\sigma_j(t)$  are complex-valued functions of bounded variations and are continuous from the right for  $t > 0$ . There exist finite limits  $H_j := \sigma_j(+0) - \sigma_j(0)$ . Linear forms (2) can be written in the form

$$U_j(y) := H_j y(0) + \int_0^T y(t) d\sigma_{j0}(t), \quad j = 1, 2, \quad (3)$$

where  $\sigma_{j0}(t)$  in (3) are complex-valued functions of bounded variations and are continuous from the right for  $t \geq 0$ . For definiteness we assume that  $H_1 \neq 0$ .

Boundary problems with nonlocal conditions are a part of fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology and etc. Nonlocal conditions come up when value of the function on the boundary is connected to values inside the domain. Theoretical investigation of problems with various type of nonlocal boundary conditions is actual problem

and recently it is paid much attention for them in the literature. Originators of such problems were Samarskii and Bitsadze. They formulated and investigated nonlocal boundary problem for elliptic equation [1]. Afterwards the number of differential problems with nonlocal boundary conditions had increased.

Quite new area, related to problems of this type, deals with investigation of the spectrum of differential equations with nonlocal conditions. In this paper we study inverse spectral problems for Sturm-Liouville operators with nonlocal boundary conditions, which are defined with the help of linear forms (2) or (3). Classical inverse problems for Eq. (1) with two-point separated boundary conditions have been studied fairly completely in many works (see the monographs [2]-[5] and the references therein). The theory of nonlocal inverse spectral problems now is only at the beginning because of its complexity. Some aspects of the inverse problem theory for different nonlocal operators can be found in [6]-[15]. In this paper we prove uniqueness theorems for the solution of the inverse spectral problems for Eq. (1) with nonlocal boundary conditions. In Section 1 we suggest statements of the inverse problems and formulate our main results (Theorems 1 and 2). Section 2 introduces important notions and properties of spectral characteristics. The proofs of Theorems 1 and 2 are given in Section 3. For this purpose we use the ideas of the method of spectral mappings [5]. In Section 4 we present counterexamples related to the statements of the inverse problems (see also [8]). Additional spectral data are introduced in Section 5. In Section 6, as an example, we consider the inverse problem of recovering the potential  $q$  from the given three spectra.

Let  $X_k(x, \lambda)$  and  $Z_k(x, \lambda)$ ,  $k = 1, 2$ , be the solutions of Eq. (1) under the initial conditions

$$X_1(0, \lambda) = X_2'(0, \lambda) = Z_1(T, \lambda) = Z_2'(T, \lambda) = 1,$$

$$X_1'(0, \lambda) = X_2(0, \lambda) = Z_1'(T, \lambda) = Z_2(T, \lambda) = 0.$$

Consider the boundary value problem (BVP)  $L_0$  for Eq. (1) with the conditions

$$U_1(y) = U_2(y) = 0.$$

Denote  $\omega(\lambda) := \det[U_j(X_k)]_{j,k=1,2}$ , and assume that  $\omega(\lambda) \not\equiv 0$ . The function  $\omega(\lambda)$  is entire in  $\lambda$  of order  $1/2$ , and its zeros  $\Xi = \{\xi_n\}_{n \geq 1}$  coincide with the eigenvalues of  $L_0$ . The function  $\omega(\lambda)$  is called the characteristic function for  $L_0$ .

Denote  $V_j(y) := y^{(j-1)}(T)$ ,  $j = 1, 2$ . Consider the BVP  $L_j$ ,  $j = 1, 2$ , for Eq. (1) with the conditions  $U_j(y) = V_1(y) = 0$ . The eigenvalues  $\Lambda_j = \{\lambda_{nj}\}_{n \geq 1}$  of the BVP  $L_j$  coincide with the zeros of the characteristic function  $\Delta_j(\lambda) := \det[U_j(X_k), V_1(X_k)]_{k=1,2}$ .

Let  $\Phi(x, \lambda)$  be the solution of Eq. (1) under the conditions  $U_1(\Phi) = 1$ ,  $V_1(\Phi) = 0$ . Denote  $M(\lambda) := U_2(\Phi)$ . The function  $M(\lambda)$  is called the Weyl-type function. It is known [4] that for Sturm-Liouville operators with classical two-point separated

boundary conditions, the specification of the Weyl function uniquely determines the potential  $q(x)$ . In our case with nonlocal boundary conditions, it is not true; the specification of the Weyl-type function  $M(\lambda)$  does not uniquely determine the potential (see counterexamples in Section 4). In our case the inverse problem is formulated as follows.

**Inverse problem 1.** Let  $\Lambda_1 \cap \Xi = \emptyset$  (condition  $S$ ). Given  $M(\lambda)$  and  $\omega(\lambda)$ , construct the potential  $q(x)$ .

We note that the functions  $\sigma_j(t)$  are known a priori, and only the potential  $q(x)$  has to be constructed.

Let us formulate a uniqueness theorem for Inverse problem 1. For this purpose, together with  $q$  we consider another potential  $\tilde{q}$ , and we agree that if a certain symbol  $\alpha$  denotes an object related to  $q$ , then  $\tilde{\alpha}$  will denote an analogous object related to  $\tilde{q}$ .

**Theorem 1.** Let  $\Lambda_1 \cap \Xi = \emptyset$ . If  $M(\lambda) = \tilde{M}(\lambda)$  and  $\omega(\lambda) = \tilde{\omega}(\lambda)$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ .

Thus, under condition  $S$ , the specification  $M(\lambda)$  and  $\omega(\lambda)$  uniquely determines the potential. The proof of Theorem 1 see below in Section 3. We note that if condition  $S$  does not hold, then the specification  $M(\lambda)$  and  $\omega(\lambda)$  does not uniquely determine the potential (see counterexamples in Section 4). In this case we have to specify an additional spectral information (see Section 5).

Consider the BVP  $L_{11}$  for Eq. (1) with the conditions  $U_1(y) = V_2(y) = 0$ . The eigenvalues  $\Lambda_{11} := \{\lambda_{n1}^1\}_{n \geq 1}$  of the BVP  $L_{11}$  coincide with the zeros of the characteristic function  $\Delta_{11}(\lambda) := \det[U_1(X_k), V_2(X_k)]_{k=1,2}$ . Clearly,  $\{\lambda_{n1}\}_{n \geq 1} \cap \{\lambda_{n,1}^1\}_{n \geq 1} = \emptyset$ .

**Inverse problem 2.** Given  $\{\lambda_{n1}, \lambda_{n1}^1\}_{n \geq 1}$ , construct  $q(x)$ .

This inverse problem is a generalization of the well-known Borg's inverse problem [16] for Sturm-Liouville operators with classical two-point separated boundary conditions, and coincides with it when  $U_1(y) = y(0)$ . We note that in Inverse problem 2 there are no restrictions on behavior of the spectra.

**Theorem 2.** If  $\lambda_{n1} = \tilde{\lambda}_{n1}, \lambda_{n1}^1 = \tilde{\lambda}_{n1}^1, n \geq 1$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ .

The proof of Theorem 2 see below in Section 3.

**2. Auxiliary propositions.** Let  $\lambda = \rho^2, \tau := \text{Im } \rho \geq 0$ . It is known (see, for example, [4]) that there exists a fundamental system of solutions  $\{Y_k(x, \rho)\}_{k=1,2}$  of Eq. (1) such that for  $|\rho| \rightarrow \infty$ :

$$Y_1^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x)(1 + O(\rho^{-1})), \quad Y_2^{(\nu)}(x, \rho) = (-i\rho)^\nu \exp(-i\rho x)(1 + O(\rho^{-1})), \quad (4)$$

$$\det[Y_k^{(\nu-1)}(x, \rho)]_{k,\nu=1,2} = -2i\rho(1 + O(\rho^{-1})). \quad (5)$$

**Lemma 1.** *Let  $\{W_k(x, \lambda)\}_{k=1,2}$  be a fundamental system of solutions of Eq. (1), and let  $Q_j(y)$ ,  $j = 1, 2$ , be linear forms. Then*

$$\det[Q_j(W_k)]_{k,j=1,2} = \det[Q_j(X_k)]_{k,j=1,2} \det[W_k^{(\nu-1)}(x, \lambda)]_{k,\nu=1,2}. \quad (6)$$

*Proof.* One has for  $\nu = 1, 2$ ,

$$W_\nu(x, \lambda) = \sum_{k=1}^2 A_{\nu k}(\lambda) X_k(x, \lambda),$$

where the coefficients  $A_{\nu k}(\lambda)$  do not depend on  $x$ . This yields

$$\det[Q_j(W_k)]_{k,j=1,2} = \det[Q_j(X_k)]_{k,j=1,2} \det[A_{\nu k}(\lambda)]_{k,j=1,2},$$

and

$$\det[W_k^{(\nu-1)}(x, \lambda)]_{k,\nu=1,2} = \det[X_k^{(\nu-1)}(x, \lambda)]_{k,\nu=1,2} \det[A_{\nu k}(\lambda)]_{k,j=1,2}.$$

Since  $\det[X_k^{(\nu-1)}(x, \lambda)]_{k,\nu=1,2} = 1$ , we arrive at (6).  $\square$

It follows from (5)-(6) that

$$\det[Q_j(Z_k)]_{k,j=1,2} = \det[Q_j(X_k)]_{k,j=1,2}, \quad (7)$$

$$\det[Q_j(Y_k)]_{k,j=1,2} = -2i\rho(1 + O(\rho^{-1})) \det[Q_j(X_k)]_{k,j=1,2}. \quad (8)$$

Consider the functions

$$\varphi(x, \lambda) = -\det[X_k(x, \lambda), U_1(X_k)]_{k=1,2}, \quad \theta(x, \lambda) = \det[X_k(x, \lambda), U_2(X_k)]_{k=1,2},$$

$$\psi(x, \lambda) = \det[X_k(x, \lambda), V_1(X_k)]_{k=1,2}.$$

Clearly,

$$U_1(\varphi) = 0, \quad U_2(\varphi) = \omega(\lambda), \quad V_1(\varphi) = \Delta_1(\lambda), \quad V_2(\varphi) = \Delta_{11}(\lambda),$$

$$U_1(\theta) = \omega(\lambda), \quad U_2(\theta) = 0, \quad V_1(\theta) = -\Delta_2(\lambda),$$

$$U_j(\psi) = \Delta_j(\lambda), \quad V_1(\psi) = 0, \quad V_2(\psi) = -1.$$

Moreover, using (6)-(7), we calculate

$$\det[\theta^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = \omega(\lambda), \quad \det[\psi^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = \Delta_1(\lambda), \quad (9)$$

$$\Delta_1(\lambda) = -U_1(Z_2), \quad \Delta_2(\lambda) = -U_2(Z_1), \quad \Delta_{11}(\lambda) = U_1(Z_1). \quad (10)$$

Comparing boundary conditions on  $\Phi, \psi, \varphi$  and  $\theta$ , we obtain

$$\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta_1(\lambda)}, \quad (11)$$

$$\Phi(x, \lambda) = \frac{1}{\omega(\lambda)} \left( \theta(x, \lambda) + \frac{\Delta_2(\lambda)}{\Delta_1(\lambda)} \varphi(x, \lambda) \right). \quad (12)$$

Hence,

$$M(\lambda) := U_2(\Phi) = \frac{\Delta_2(\lambda)}{\Delta_1(\lambda)}, \quad (13)$$

$$\det[\Phi^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = 1. \quad (14)$$

Let  $v_1(x, \lambda)$  and  $v_2(x, \lambda)$  be the solutions of Eq. (1) under the conditions

$$v_1(T, \lambda) = v_2'(T, \lambda) = 1, \quad v_1'(T, \lambda) = 0, \quad U_1(v_2) = 0.$$

Obviously,

$$v_1(x, \lambda) = Z_1(x, \lambda), \quad v_2(x, \lambda) = Z_2(x, \lambda) + N(\lambda)Z_1(x, \lambda), \quad \det[v_k^{(\nu-1)}(x, \lambda)]_{k,\nu=1,2} = 1, \quad (15)$$

where

$$N(\lambda) = \frac{\Delta_1(\lambda)}{\Delta_{11}(\lambda)} = -\frac{U_1(Z_2)}{U_1(Z_1)}. \quad (16)$$

Denote

$$U_1^a(y) := \int_0^a y(t) d\sigma_1(t), \quad a \in (0, T].$$

Clearly,  $U_1 = U_1^T$ , and if  $\sigma_1(t) \equiv C$  (constant) for  $t \geq a$ , then  $U_1 = U_1^a$ .

Let  $\lambda_{n1} = \rho_n^2$ . For sufficiently small  $\delta > 0$ , we denote

$$\Pi_\delta := \{\rho : \arg \rho \in [\delta, \pi - \delta]\}, \quad G_\delta := \{\rho : |\rho - \rho_n| \geq \delta, \quad \forall n \geq 1\}.$$

**Lemma 2.** For  $|\rho| \rightarrow \infty$ ,  $\rho \in \Pi_\delta$ ,

$$\Phi^{(\nu)}(x, \lambda) = \frac{(i\rho)^\nu}{H_1} \exp(i\rho x)(1 + o(1)), \quad x \in [0, T], \quad (17)$$

$$v_1^{(\nu)}(x, \lambda) = \frac{(i\rho)^\nu}{2} \exp(-i\rho(T - x))(1 + O(\rho^{-1})), \quad x \in [0, T], \quad (18)$$

$$\Delta_1(\lambda) = -\frac{H_1}{2i\rho} \exp(-i\rho T)(1 + o(1)), \quad \Delta_{11}(\lambda) = \frac{H_1}{2} \exp(-i\rho T)(1 + o(1)). \quad (19)$$

Let  $\sigma_1(t) \equiv C$  (constant) for  $t \geq a$  (i.e.  $U_1 = U_1^a$ ). Then for  $|\rho| \rightarrow \infty$ ,  $\rho \in \Pi_\delta$ ,

$$\varphi^{(\nu)}(x, \lambda) = \frac{H_1}{2} (-i\rho)^{\nu-1} \exp(-i\rho x)(1 + o(1) + O(\exp(i\rho(2x - a)))), \quad x \in (0, T], \quad (20)$$

$$v_2^{(\nu)}(x, \lambda) = (-i\rho)^{\nu-1} \exp(i\rho(T-x))(1+o(1)+O(\exp(i\rho(2x-a)))), \quad x \in [0, T]. \quad (21)$$

*Proof.* One has

$$\Phi(x, \lambda) = A_1(\lambda)Y_1(x, \rho) + A_2(\lambda)Y_2(x, \rho). \quad (22)$$

Since  $U_1(\Phi) = 1$ ,  $V_1(\Phi) = 0$ , it follows from (22) that

$$A_1(\lambda)U_1(Y_1) + A_2(\lambda)U_1(Y_2) = 1, \quad A_1(\lambda)V_1(Y_1) + A_2(\lambda)V_1(Y_2) = 0. \quad (23)$$

By virtue of (4), we have for  $|\rho| \rightarrow \infty$ ,  $\rho \in \Pi_\delta$ :

$$U_1(Y_1) = H_1(1 + o(1)), \quad U_1(Y_2) = O(\exp(-i\rho T)), \quad (24)$$

$$V_1(Y_1) = \exp(i\rho T)(1 + O(\rho^{-1})), \quad V_1(Y_2) = \exp(-i\rho T)(1 + O(\rho^{-1})). \quad (25)$$

Solving linear algebraic system (23) and using (24)-(25), we calculate

$$A_1(\rho) = H_1^{-1}(1 + o(1)), \quad A_2(\rho) = O(\exp(2i\rho T)).$$

Substituting these relations into (22), we arrive at (17). Formulas (18)-(21) are proved similarly.  $\square$

By the well-known method (see, for example, [4]) one can also obtain the following estimates for  $x \in (0, T)$ ,  $\tau \geq 0$ :

$$v_1^{(\nu)}(x, \lambda) = O(\rho^\nu \exp(-i\rho(T-x))), \quad (26)$$

$$\Phi^{(\nu)}(x, \lambda) = O(\rho^\nu \exp(i\rho x)), \quad \rho \in G_\delta. \quad (27)$$

Moreover, if  $\sigma_1(t) \equiv C$  (constant) for  $t \geq a$  (i.e.  $U_1 = U_1^a$ ), then for  $x \geq a/2$ ,  $\tau \geq 0$ :

$$\varphi^{(\nu)}(x, \lambda) = O(\rho^{\nu-1} \exp(-i\rho x)), \quad (28)$$

$$v_2^{(\nu)}(x, \lambda) = O(\rho^{\nu-1} \exp(i\rho(T-x))), \quad \rho \in G_\delta. \quad (29)$$

**3. Proofs of Theorems 1-2.** Firstly we prove Theorem 2. Let  $\lambda_{n1} = \tilde{\lambda}_{n1}, \lambda_{n1}^1 = \tilde{\lambda}_{n1}^1$ ,  $n \geq 1$ . The characteristic function  $\Delta_1(\lambda)$  of the BVP  $L_1$  is entire in  $\lambda$  of order  $1/2$ . Therefore, by Hadamard's factorization theorem,  $\Delta_1(\lambda)$  is uniquely determined up to a multiplicative constant by its zeros, i.e.  $\Delta_1(\lambda)/\tilde{\Delta}_1(\lambda) \equiv C$  (constant). Taking (19) into account, we calculate  $C = 1$ , and consequently,  $\Delta_1(\lambda) \equiv \tilde{\Delta}_1(\lambda)$ . Analogously, we get  $\Delta_{11}(\lambda) \equiv \tilde{\Delta}_{11}(\lambda)$ . By virtue of (16), this yields

$$N(\lambda) \equiv \tilde{N}(\lambda). \quad (30)$$

Consider the functions

$$P_1(x, \lambda) = v_1(x, \lambda)\tilde{v}'_2(x, \lambda) - \tilde{v}'_1(x, \lambda)v_2(x, \lambda), \quad P_2(x, \lambda) = v_2(x, \lambda)\tilde{v}_1(x, \lambda) - \tilde{v}_2(x, \lambda)v_1(x, \lambda). \quad (31)$$

In view of (15) and (30), one gets

$$\begin{aligned} P_1(x, \lambda) &= (Z_1(x, \lambda)\tilde{Z}'_2(x, \lambda) - \tilde{Z}'_1(x, \lambda)Z_2(x, \lambda)) + (\tilde{N}(\lambda) - N(\lambda))Z_1(x, \lambda)\tilde{Z}'_1(x, \lambda) \\ &= Z_1(x, \lambda)\tilde{Z}'_2(x, \lambda) - \tilde{Z}'_1(x, \lambda)Z_2(x, \lambda), \\ P_2(x, \lambda) &= Z_2(x, \lambda)\tilde{Z}_1(x, \lambda) - \tilde{Z}_2(x, \lambda)Z_1(x, \lambda) + (N(\lambda) - \tilde{N}(\lambda))Z_1(x, \lambda)\tilde{Z}_1(x, \lambda) \\ &= Z_2(x, \lambda)\tilde{Z}_1(x, \lambda) - \tilde{Z}_2(x, \lambda)Z_1(x, \lambda). \end{aligned}$$

Thus, for each fixed  $x$ , the functions  $P_k(x, \lambda)$ ,  $k = 1, 2$ , are entire in  $\lambda$ . On the other hand, taking (18) and (21) into account we calculate for each fixed  $x \geq T/2$  and  $k = 1, 2$ :

$$P_k(x, \lambda) - \delta_{1k} = o(1), \quad |\rho| \rightarrow \infty, \quad \rho \in \Pi_\delta,$$

where  $\delta_{1k}$  is the Kronecker symbol. Moreover, in view of (26) and (29), we get for  $k = 1, 2$

$$P_k(x, \lambda) = O(1), \quad |\rho| \rightarrow \infty, \quad \rho \in \Pi_\delta.$$

Using the maximum modulus principle and Liouville's theorem for entire functions, we conclude that

$$P_1(x, \lambda) \equiv 1, \quad P_2(x, \lambda) \equiv 0, \quad x \geq T/2.$$

Together with (31) this yields

$$v_k(x, \lambda) = \tilde{v}_k(x, \lambda), \quad Z_k(x, \lambda) = \tilde{Z}_k(x, \lambda), \quad q(x) = \tilde{q}(x), \quad x \geq T/2. \quad (32)$$

Let us now consider the BVPs  $L_1^a$  and  $L_{11}^a$  for Eq. (1) on the interval  $(0, T)$  with the conditions  $U_1^a(y) = V_1(y) = 0$  and  $U_1^a(y) = V_2(y) = 0$ , respectively. Then, according to (10), the functions  $\Delta_1^a(\lambda) := -U_1^a(Z_2)$  and  $\Delta_{11}^a(\lambda) := U_1^a(Z_1)$  are the characteristic functions of  $L_1^a$  and  $L_{11}^a$ , respectively. One has

$$U_1^{a/2}(Z_k) = U_1^a(Z_k) - \int_{a/2}^a Z_k(t, \lambda) d\sigma_1(t), \quad k = 1, 2,$$

hence

$$\Delta_1^{a/2}(\lambda) = \Delta_1^a(\lambda) + \int_{a/2}^a Z_2(t, \lambda) d\sigma_1(t), \quad \Delta_{11}^{a/2}(\lambda) = \Delta_{11}^a(\lambda) - \int_{a/2}^a Z_1(t, \lambda) d\sigma_1(t). \quad (33)$$

Let us use (33) for  $a = T$ . Since  $\Delta_1^T(\lambda) = \Delta_1(\lambda)$ ,  $\Delta_{11}^T(\lambda) = \Delta_{11}(\lambda)$ , it follows from (32)-(33) that

$$\Delta_1^{T/2}(\lambda) = \tilde{\Delta}_1^{T/2}(\lambda), \quad \Delta_{11}^{T/2}(\lambda) = \tilde{\Delta}_{11}^{T/2}(\lambda).$$

Repeating preceding arguments subsequently for  $a = T/2, T/4, T/8, \dots$ , we conclude that  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ . Theorem 2 is proved.  $\square$

Now we will prove Theorem 1. Let  $\Lambda_1 \cap \Xi = \emptyset$ , and let  $M(\lambda) = \tilde{M}(\lambda)$ ,  $\omega(\lambda) = \tilde{\omega}(\lambda)$ . Consider the functions

$$R_1(x, \lambda) = \Phi(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{\Phi}'(x, \lambda)\varphi(x, \lambda), \quad R_2(x, \lambda) = \varphi(x, \lambda)\tilde{\Phi}(x, \lambda) - \tilde{\varphi}(x, \lambda)\Phi(x, \lambda). \quad (34)$$

It follows from (11) and (34) that

$$R_1(x, \lambda) = \frac{1}{\Delta_1(\lambda)} \left( \psi(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{\psi}'(x, \lambda)\varphi(x, \lambda) \right),$$

$$R_2(x, \lambda) = \frac{1}{\Delta_1(\lambda)} \left( \varphi(x, \lambda)\tilde{\psi}(x, \lambda) - \tilde{\varphi}(x, \lambda)\psi(x, \lambda) \right).$$

This yields that for each fixed  $x$ , the functions  $R_k(x, \lambda)$  are meromorphic in  $\lambda$  with possible poles only at  $\lambda = \lambda_{n1}$ . On the other hand, taking (12) into account we calculate

$$R_1(x, \lambda) = \frac{1}{\omega(\lambda)} \left( \theta(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{\theta}'(x, \lambda)\varphi(x, \lambda) \right), \quad (35)$$

$$R_2(x, \lambda) = \frac{1}{\omega(\lambda)} \left( \varphi(x, \lambda)\tilde{\theta}(x, \lambda) - \tilde{\varphi}(x, \lambda)\theta(x, \lambda) \right). \quad (36)$$

Hence the functions  $R_k(x, \lambda)$  are regular at  $\lambda = \lambda_{n1}$ . Thus, for each fixed  $x$ , the functions  $R_k(x, \lambda)$  are entire in  $\lambda$ . Furthermore, by virtue of (17) and (20), we obtain for  $x \geq T/2$ :

$$R_k(x, \lambda) - \delta_{1k} = o(1), \quad |\rho| \rightarrow \infty, \quad \rho \in \Pi_\delta.$$

Moreover, using (27)-(28), we get for  $x \geq T/2$ :

$$R_k(x, \lambda) = O(1), \quad |\rho| \rightarrow \infty, \quad \rho \in G_\delta.$$

Therefore,  $R_1(x, \lambda) \equiv 1$ ,  $R_2(x, \lambda) \equiv 0$ . Together with (14) and (34), this yields

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda), \quad \psi(x, \lambda) = \tilde{\psi}(x, \lambda), \quad q(x) = \tilde{q}(x), \quad x \geq T/2.$$

In particular, we obtain

$$Z_k(x, \lambda) = \tilde{Z}_k(x, \lambda), \quad k = 1, 2, \quad x \geq T/2.$$

Since

$$\varphi(x, \lambda) = U_1(Z_1)Z_2(x, \lambda) - U_1(Z_2)Z_1(x, \lambda),$$



it follows that

$$\Delta_1(\lambda) = \tilde{\Delta}_1(\lambda), \Delta_{11}(\lambda) = \tilde{\Delta}_{11}(\lambda).$$

Using Theorem 2, we conclude that  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ . Theorem 1 is proved.  $\square$

**4. Counterexamples.** 1) Let  $T = \pi$ ,  $U_1(y) = y(0)$ ,  $U_2(y) = y(\pi/2)$ ,  $q(x) = q(x + \pi/2)$ ,  $x \in (0, \pi/2)$ , and  $q(x) \not\equiv q(\pi - x)$ . Take  $\tilde{q}(x) = q(\pi - x)$ ,  $x \in (0, \pi)$ . We see that the BVP  $\tilde{L}_1$ , for Eq. (1) with  $\tilde{q}(x) = q(\pi - x)$  and the conditions  $U_1(y) = V_1(y) = 0$ ; the BVP  $\tilde{L}_2$ , for Eq. (1) with  $\tilde{q}(x) = q(\pi - x)$  and the conditions  $U_2(y) = V_1(y) = 0$ ; and the BVP  $\tilde{L}_0$ , for Eq. (1) with  $\tilde{q}(x) = q(\pi/2 - x)$  and the conditions  $U_1(y) = U_2(y) = 0$ .

Then

$$\Delta_1(\lambda) = \tilde{\Delta}_1(\lambda), \Delta_2(\lambda) = \tilde{\Delta}_2(\lambda), \omega(\lambda) = \tilde{\omega}(\lambda),$$

and, in view of (13),  $M(\lambda) = \tilde{M}(\lambda)$ . Condition  $S$  does not hold. This means, that the specification of  $M(\lambda)$  and  $\omega(\lambda)$  does not uniquely determine the potential  $q$ .

2) Let  $T = \pi$ ,  $U_1(y) = y(0)$ ,  $U_2(y) = y(\pi - \alpha)$ , where  $\alpha \in (0, \pi/2)$ . Then

$$\Delta_1(\lambda) = X_2(\pi, \lambda), \omega(\lambda) = X_2(\pi - \alpha, \lambda),$$

and

$$\Delta_2(\lambda) = X_2(\pi, \lambda)X_1(\pi - \alpha, \lambda) - X_2(\pi - \alpha, \lambda)X_1(\pi, \lambda).$$

Obviously, if  $\Delta_1(\lambda^*)\Delta_2(\lambda^*)\omega(\lambda^*) = 0$  for a certain  $\lambda^*$ , then either  $\Delta_1(\lambda^*) = \Delta_2(\lambda^*) = \omega(\lambda^*) = 0$  (i.e.  $\lambda^*$  is an eigenvalue for all boundary value problems  $L_0, L_1, L_2$ ), or  $\lambda^*$  is an eigenvalue for only one problem from  $L_0, L_1, L_2$ . In other words, it is impossible that  $\lambda^*$  is an eigenvalue for only two problems from  $L_0, L_1, L_2$ .

Let  $q(x) \not\equiv q(\pi - x)$ , and let  $q(x) \equiv 0$  for  $x \in [0, \alpha_0] \cup [\pi - \alpha_0, \pi]$ , where  $\alpha_0 \in (0, \pi/2)$ . If  $\alpha < \alpha_0$ , then  $\lambda_{n2} = (\pi n / \alpha)^2$ ,  $n \geq 1$ . Choose a sufficiently small  $\alpha < \alpha_0$  such that  $\Lambda_1 \cap \Lambda_2 = \emptyset$ . Clearly, such choice is possible. Then  $\Lambda_1 \cap \Xi = \emptyset$ , i.e. condition  $S$  holds. Take  $\tilde{q}(x) := q(\pi - x)$ . Then  $\Delta_1(\lambda) = \tilde{\Delta}_1(\lambda)$ ,  $\Delta_2(\lambda) = \tilde{\Delta}_2(\lambda)$ , and consequently,  $M(\lambda) = \tilde{M}(\lambda)$ . Thus, condition  $S$  holds, but the specification of  $M(\lambda)$  does not uniquely determine the potential  $q$ .

**5. Additional spectral data.** If condition  $S$  does not hold, then the specification of  $M(\lambda)$  and  $\omega(\lambda)$  does not uniquely determine the potential  $q$ . We introduce additional spectral data. For simplicity, we confine ourselves to the case when zeros of  $\omega(\lambda)$  are simple. By virtue of (9),

$$\det[\theta^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = \omega(\lambda).$$

Then the functions  $\varphi(x, \xi_n)$  and  $\theta(x, \xi_n)$  are linearly dependent, i.e. there exist numbers  $A_n$  and  $B_n$  ( $|A_n| + |B_n| > 0$ ) such that  $A_n\varphi(x, \xi_n) = B_n\theta(x, \xi_n)$ .

Consider the sequence  $D = \{d_n\}_{n \geq 1}$ , where  $d_n := B_n/A_n$  ( $d_n := \infty$  if  $A_n = 0$ ). The inverse problem is formulated as follows.

**Inverse problem 3.** Given  $M(\lambda), \omega(\lambda)$  and  $D$ , construct  $q(x)$ .

We note that if condition  $S$  holds (i.e.  $\Lambda_1 \cap \Xi = \emptyset$ ), then by virtue of (11)-(12),  $d_n = -M^{-1}(\xi_n)$ . At this case  $M(\lambda) = \tilde{M}(\lambda)$  implies that  $D = \tilde{D}$ , and we arrive at Inverse problem 1.

**Theorem 3.** If  $M(\lambda) = \tilde{M}(\lambda)$ ,  $\omega(\lambda) = \tilde{\omega}(\lambda)$  and  $D = \tilde{D}$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ .

*Proof.* Fix  $n \geq 1$ , and consider the functions  $R_k(x, \lambda)$ ,  $k = 1, 2$ , in a neighborhood of the point  $\lambda = \xi_n$ . If  $d_n = \tilde{d}_n \neq \infty$ , then, in view of (35)-(36),  $R_k(x, \lambda)$  are regular at  $\lambda = \xi_n$ . If  $d_n = \tilde{d}_n = \infty$ , then  $\theta(x, \xi_n) = \tilde{\theta}(x, \xi_n) = 0$ . By virtue of (35)-(36), this yields that  $R_k(x, \lambda)$  are regular at  $\lambda = \xi_n$ . Thus, the functions  $R_k(x, \lambda)$ ,  $k = 1, 2$  are entire in  $\lambda$ . Repeating the arguments from the proof of Theorem 1, we conclude that  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ .  $\square$

**6. Example (Inverse problem from three spectra).** Fix  $a \in (0, T)$ . Consider Inverse problem 1 in the particular case, when  $U_1(y) = y(0)$ ,  $U_2(y) = y(a)$ . Then the boundary value problems  $L_0, L_1, L_2$  take the form

$$L'_0 : \quad y(0) = y(a) = 0,$$

$$L'_1 : \quad y(0) = y(T) = 0,$$

$$L'_2 : \quad y(a) = y(T) = 0.$$

Denote by  $\Lambda'_j = \{\lambda'_{nj}\}$  the spectrum of  $L'_j$ , and assume that  $\Lambda'_0 \cap \Lambda'_1 = \emptyset$  (condition  $S'$ ).

**Inverse problem 4.** Given three spectra  $\Lambda'_0, \Lambda'_1$  and  $\Lambda'_2$ , construct  $q(x)$ .

The following theorem is a consequence of Theorem 1.

**Theorem 2.** Let condition  $S'$  hold. If  $\Lambda'_j = \tilde{\Lambda}'_j$ ,  $j = 0, 1, 2$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ .

We note that Inverse problem 4 was studied by many authors (see, for example, [17]-[18]).

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